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Projectors for the fuzzy sphere

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Abstract

All fiber bundle with a given set of characteristic classes are viewable as particular projections of a more general bundle called a universal classifying space. This notion of projector valued field, a global definition of connections and gauge fields, can be useful to define vector bundles for non commutative base spaces. In this paper we derive the projector valued field for the fuzzy sphere, defining non-commutative n-monopole configurations, and check that in the classical limit, using the machinery of non-commutative geometry, the corresponding topological charges (Chern class) are integers.

1 Introduction

In classical gauge theory, the basic variables like connections and gauge fields are usually defined locally in space. There is however a way to descrive gauge theory by globally defined fields, using the fiber bundle formulation. These globally defined fields are called projectors and serve to give an alternative description to the connections which is suitable for generalizations like non-commutative geometry [1]-[2]. The theorem of Narasimham and Ramanan shows that all fiber bundles with a given set of characteristic classes are viewable as particular projections of a more general bundle called a universal space bundle. Consider a fiber bundle determined by the unitary groups $U(p)$ as gauge groups and by basic space a (compcat) manifold denoted by M. A principal $U(p)$ bundle E over M with a connection form ω can be associated to a projector valued field $P(x)(x \in M)$. In more general terms, the space $\Gamma(M, E)$ of smooth sections of the vector bundle $E \rightarrow M$ over a compact manifold M is a finite projective module over the commutative algebra $C(M)$ and every finite projective $C(M)$-module can be realized as the module of sections of some vector bundle over M. The correspondence is quite general and well known in the mathematical literature, less known between the physicist community.

This equivalence has received particular attention in generalizing the classical geometry to non commutative geometry, and is the nowadays basis to generalize the concept of vector bundles to the noncommutative case [3]-[4]. In ref. [5] a finite-projective-module description of all monopoles configurations on the 2-dimensional sphere S^2 has been presented. In this paper we outline its generalization to one of the simplest example of non-commutative manifolds, the fuzzy sphere [6]-[7]. We find that it is simple to give the description of the non commutative n-monopoles over the fuzzy sphere in terms of $(|n|+1)$ x $(|n|+1)$ matrices, having as entries the elements of the basic non commutative coordinate algebra of the fuzzy sphere.

An alternative, quite involved, formulation of the projectors has been given in reference [8], and resembles the classical description of the projectors over S^2 of ref. [9]. Our formulation is surely original, more easy to deal with and should have all the good properties of the analogous one over S^2 , proposed by Landi.

As an application of our $n-monopoles$ projectors, we show how to compute their topological charge (the Chern class) in the classical limit, using the machinery of non commutative geometry, i.e. the classical Dirac operator introduced in [10].

2 Fuzzy sphere

The fuzzy sphere is defined through an algebra M_n of n x n complex matrices. When $n \to \infty$ the corresponding matrix geometry tends to the geometry of the 2-dimesional sphere. In order to realize such an idea, one views the ordinary geometry of the sphere as a commutative algebra. A process of truncations of it produces the matrix algebras, non-commutative algebras which approximate the commutative algebra in the large n limit.

The S^2 sphere can be defined as the constraint between the coordinates:

$$g_{ij}x^i x^j = r^2 \tag{2.1}$$

The geometry of S^2 can be formulated in terms of the algebra $\mathcal{C}(S^2)$ of functions $f(x^i)$ on S^2 admitting a polynomial expansion in the x^i:

$$f(x^i) = f_0 + f_i x^i + \frac{1}{2} f_{ij} x^i x^j + ... \tag{2.2}$$

Let us truncate this algebra by requiring that all coefficients in the expansion apart from f_0 and f_i are zero. In order to turn this linear approximation to a true algebra, it is possible to replace the x^i with the algebra M_2 of complex 2 x 2 complex matrices. Therefore we identify x^i with the algebra of Pauli matrices:

$$x^i \to x^i = \frac{\alpha}{2}\sigma^i \tag{2.3}$$

Moreover the Casimir of the algebra can be taken as a constant and coincides with the constraint (2.1) , by imposing that

$$\alpha^2 = \frac{4}{3}r^2 \tag{2.4}$$

The algebra M_2 describes a fuzzy sphere, where only the north and south poles can be distinguished.

Keeping the term quadratic in the x^i, the quadratic approximation to the commutative algebra of functions can be turned into an algebra, by identifying x^i with the algebra M_3 of complex 3 x 3 matrices. Generalizing this truncation to the n-term of the series (2.2) we can define the general fuzzy sphere in terms of the algebra M_n of complex n x n matrices, through the identification

$$x^i \to x^i = \alpha J^i \tag{2.5}$$

where J^i is a n x n representation of the $SU(2)$ algebra.

Therefore the fuzzy sphere [6]-[7]-[10]-[11]-[12]-[13] can be defined by the commutation relations:

$$\begin{aligned} [x_i, x_j] &= i\alpha\epsilon_{ijk}x_k \\ \alpha &= \frac{r}{\sqrt{j(j+1)}} \\ \sum_i (x^i)^2 &= r^2 \end{aligned} \tag{2.6}$$

This can be realized as an algebra of a couple of oscillators

$$\begin{aligned} x_1 &= \frac{\hat{\alpha}}{2}(a_0 a_1^\dagger + a_1 a_0^\dagger) \\ x_2 &= \frac{\hat{\alpha}}{2} i(a_0 a_1^\dagger - a_1 a_0^\dagger) \\ x_3 &= \frac{\hat{\alpha}}{2}(a_0^\dagger a_0 - a_1^\dagger a_1) \\ \hat{N} &= a_0^\dagger a_0 + a_1^\dagger a_1 \end{aligned} \tag{2.7}$$

together with the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{2.8}$$

where the operator $\hat{\alpha}$ is defined as

$$\hat{\alpha} = 2\frac{r}{\sqrt{\hat{N}(\hat{N}+2)}} \tag{2.9}$$

and is equal to α for representations at fixed number $\hat{N} = N = 2j$, as we demand to build our fixed-j fuzzy sphere.

In the classical limit $N \to \infty$ this construction corresponds to the $U(1)$ principal (Hopf) fibration $\pi : S^3 \to S^2$ over the two dimensional sphere S^2 [5]. An ambiguity of a $U(1)$ factor into the definition of the oscillators is cancelled in the x^i combinations. The same trick is useful to compute the projectors in terms of vector valued fields. They carry the $U(1)$ ambiguity, which is cancelled in the combination of bra and ket vectors giving rise to the projectors. Therefore our vector fields will be defined in terms of the oscillators while the projectors will be functions only of the x_i algebra.

3 The projectors for n-monopoles

The monopole connections can be alternatively described by some projected valued fields $P_n(x)$. The canonical connection associated with the projector $P_n(x)$ has curvature given by

$$\nabla^2 = P_n dP_n dP_n \tag{3.1}$$

To construct the n-monopole projectors $P_n(x)$ for the fuzzy sphere consider the n-dimensional vectors

$$|\psi_n >= N_n \begin{pmatrix} (a_0)^n \\ ... \\ \sqrt{\begin{pmatrix} n \\ k \end{pmatrix}} (a_0)^{n-k} (a_1)^k \\ ... \\ (a_1)^n \end{pmatrix} \tag{3.2}$$

The normalization condition for these vectors fixes the function N_n to be dependent on the number operator $\hat{N} = a_0^\dagger a_0 + a_1^\dagger a_1$:

$$\begin{aligned} < \psi_n|\psi_n > &= 1 \\ N_n = N_n(\hat{N}) &= \frac{1}{\sqrt{\prod_{i=0}^{n-1}(\hat{N} - i + n)}} \end{aligned} \tag{3.3}$$

The corresponding n-monopole connection 1-form has a very simple expression in terms of the vector valued function $|\psi_n >$:

$$A_n^\nabla =< \psi_n|d\psi_n > \tag{3.4}$$

The projector for the n-monopole is defined to be:

$$P_n = |\psi_n >< \psi_n| \qquad n < N+2 \tag{3.5}$$

and satisfies the basic properties of a projector, being

$$P_n^2 = P_n \qquad P_n^\dagger = P_n \tag{3.6}$$

It is easy to notice that in the product there appears only combinations of oscillators that can be written in terms of the algebra x_i of the fuzzy sphere. Therefore P_n is a matrix having as entries the basic operator algebra of the theory.

Consistently the projector P_n has a nonvanishing positive trace given by:

$$\begin{aligned} TrP_n &= Tr|\psi_n >< \psi_n| = N_n \prod_{i=0}^{n-1} (a_0^\dagger a_0 + a_1^\dagger a_1 + i + 2) N_n = \\ &= \frac{\hat{N} + n + 1}{\hat{N} + 1} \end{aligned} \tag{3.7}$$

In the classical limit $N \to \infty$ the trace is equal 1, and the corresponding classical projectors [5] are matrices of numbers of rank 1.

4 The projectors for n-antimonopoles

To construct the n-antimonopole solution it is enough to take the adjoint of the vectors (3.2). Consider the n-dimensional vectors

$$|\psi_{-n} >= N_n \begin{pmatrix} (a_0^\dagger)^n \\ \cdots \\ \sqrt{\begin{pmatrix} n \\ k \end{pmatrix}} (a_0^\dagger)^{n-k} (a_1^\dagger)^k \\ \cdots \\ (a_1^\dagger)^n \end{pmatrix} \tag{4.1}$$

The normalization condition for these vectors fixes the function N_n to be dependent on the number operator $\hat{N} = a_0^\dagger a_0 + a_1^\dagger a_1$:

$$\begin{aligned} < \psi_{-n} | \psi_{-n} > &= 1 \\ N_n = N_n(\hat{N}) &= \frac{1}{\sqrt{\prod_{i=0}^{n-1} (\hat{N} + i + 2 - n)}} \end{aligned} \tag{4.2}$$

The corresponding projector for antimonopoles is:

$$P_{-n} = |\psi_{-n} >< \psi_{-n}| \qquad n < N + 2 \tag{4.3}$$

Consistently the projector P_{-n} has a trace given by:

$$\begin{aligned} TrP_{-n} &= Tr|\psi_{-n}><\psi_{-n}| = N_n \prod_{i=0}^{n-1}(a_0^\dagger a_0 + a_1^\dagger a_1 - i)N_n = \\ &= \frac{\hat{N}+1-n}{\hat{N}+1} \end{aligned} \tag{4.4}$$

This trace is positive definite if and only if the following bound is respected:

$$n < N+1 \tag{4.5}$$

5 Classical limit and Chern Class

The projectors above, defining the n-projective moduli of the non-commutative theory, can be checked to reproduce the known results for the classical sphere (see Landi [5]). We use here the tecnology of the non-commutative geometry to compute the first Chern class, and therefore

$$c_n = \frac{1}{2\pi}\int d\Omega\ Tr\ (\gamma_5 P_n dP_n dP_n) \tag{5.1}$$

It is not difficult to manipolate the Chern class

$$P_n dP_n dP_n = |\psi_n>\ \ \{<d\psi_n|d\psi_n> + (<\psi_n|d\psi_n>)^2\}\ \ <\psi_n| \tag{5.2}$$

We use as derivation the commutator of the classical Dirac operator on the sphere [10]:

$$D = \sigma^i \cdot L^i + 1 \tag{5.3}$$

that anticommutes with the classical γ_5 operator [11] given by

$$\gamma_5 = \sigma^i \cdot \frac{x^i}{r} \tag{5.4}$$

The Lie derivative L_i has as commutation relations

$$[L^i, L^j] = i\epsilon_{ijk}L^k \tag{5.5}$$

and can be taken as the following adjoint action:

$$L^i = [\frac{x^i}{\hat{\alpha}}, .] \tag{5.6}$$

In order to compute the n-monopole 1-form connection, we first compute

$$\begin{aligned} < \psi_n | \frac{x_i}{\hat{\alpha}} | \psi_n > &= \frac{\hat{N} - n}{\hat{N}} \frac{x_i}{\hat{\alpha}} \\ < \psi_{-n} | \frac{x_i}{\hat{\alpha}} | \psi_{-n} > &= \frac{\hat{N} + 2 + n}{\hat{N} + 2} \frac{x_i}{\hat{\alpha}} \end{aligned} \tag{5.7}$$

by observing that x_i is always a combination of oscillators a_i and their adjoint $a_j^\dagger$. These can be commuted with $|\psi_{\pm n} > N_n^{-1}(\hat{N} \mp n)$ and $N_n^{-1}(\hat{N} \mp n) < \psi_{\pm n}|$ and the resulting scalar product is just the normalization condition.

Therefore the Lie derivative , applied to the oscillators, gives rise to:

$$\begin{aligned} < \psi_n | \sigma_i L_i | \psi_n > &= -\frac{n}{\hat{N}} \frac{\sigma^i x^i}{\hat{\alpha}} \\ < \psi_{-n} | \sigma_i L_i | \psi_{-n} > &= \frac{n}{\hat{N} + 2} \frac{\sigma^i x^i}{\hat{\alpha}} \end{aligned} \tag{5.8}$$

Since $(\sigma^i x^i)^2 = r^2 - \hat{\alpha}\ \sigma^i x^i$ we can compute the second term in eq. (5.2):

$$\begin{aligned} (< \psi_n | d\psi_n >)^2 &= \frac{n^2}{\hat{N}^2} (\frac{r^2}{\hat{\alpha}^2} - \frac{\sigma^i x^i}{\hat{\alpha}}) \\ (< \psi_{-n} | d\psi_{-n} >)^2 &= \frac{n^2}{(\hat{N} + 2)^2} (\frac{r^2}{\hat{\alpha}^2} - \frac{\sigma^i x^i}{\hat{\alpha}}) \end{aligned} \tag{5.9}$$

Its contribution to the first Chern class is vanishing in the classical limit, since the only surviving term to the trace is depressed by a factor $\frac{1}{N}$.

Let us compute the first term of eq. (5.2) which should give the real contribution:

$$\begin{aligned} < d\psi_n | d\psi_n > &= \\ &= \frac{\sigma^i x^i}{\hat{\alpha}} < \psi_n | \frac{\sigma^i x^i}{\hat{\alpha}} | \psi_n > - (\frac{\sigma^i x^i}{\hat{\alpha}})^2 - < \psi_n | (\frac{\sigma^i x^i}{\hat{\alpha}})^2 | \psi_n > + < \psi_n | \frac{\sigma^i x^i}{\hat{\alpha}} | \psi_n > \frac{\sigma^i x^i}{\hat{\alpha}} \\ &= n \frac{\sigma^i x^i}{N\alpha} - \frac{n(2+n)}{4} \end{aligned} \tag{5.10}$$

and

$$< d\psi_{-n}|d\psi_{-n} >= -n\frac{\sigma^i x^i}{(N+2)\alpha} - \frac{n(2+n)}{4} \tag{5.11}$$

In the classical limit the first term in r.h.s. of eqs. (5.10) and (5.11) is equivalent to $n\gamma^5/2$ while the other term is cancelled by the trace. Therefore the total contribution is given by:

$$c_n = \frac{1}{2\pi}\int d\Omega\ Tr(\gamma^5\ P_n dP_n dP_n) \ = \ n(\gamma^5)^2 \ = \ n \tag{5.12}$$

6 Conclusion

An alternative description to the connections and curvature on the fiber bundle can be obtained immerging the fiber bundle at fixed toplogy into a trivial universal bundle, from whom it can be recovered with a projection. The projected valued field give a complete global description of the vector bundle, within each topology sector.

In this paper we have shown how to derive the exact expressions for the non commutative n-monopole on the fuzzy sphere. We have checked that their topological charge (first Chern class for the vector bundle) are integers in the classical limit, using the classical expression of the Dirac operator. It would be interesting to verify that these topological charge are integers in the full non-commutative theory, but this requires the knowlegde of the full non-commutative Dirac operator on the sphere. A proposal of it is given in [12]-[13]. However, to our knowledge, it remains to be checked that it verifies all axioms of non-commutative geometry.

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